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The inviscid limit for the complex Ginzburg–Landau equation

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Abstract

We study the inviscid limit of the complex Ginzburg–Landau equation. We observe that the solutions for the complex Ginzburg–Landau equation converge to the corresponding solutions for the nonlinear Schrödinger equation. We give its convergence rate. We estimate the integral forms of solutions for two equations.

Keywords: Complex Ginzburg–Landau equation; Nonlinear Schrödinger equation; Inviscid limit; Strichartz estimate

1. Introduction

In this paper we consider the inviscid limit for the complex Ginzburg–Landau equations. We introduce two nonlinear equations, complex Ginzburg–Landau equation (CGL) and nonlinear Schrödinger equation (NLS):

$$\text{CGL:}\quad \partial_t u = (a + i\nu)\Delta u - (b + i\mu)|u|^{p-1}u, \qquad u(0) = u_0, \tag{1.1}$$

$$\text{NLS:}\quad \partial_t v = i\nu\Delta v - i\mu|v|^{p-1}v, \qquad v(0) = v_0, \tag{1.2}$$

where u and v are complex valued functions of $(x,t) \in \mathbb{R}^n \times (0,\infty)$, the parameters $p > 1$, $a > 0$, $b > 0$ and ν, μ are real. We can take $\nu > 0$ without loss of generality.

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We can see that if $a = b = 0$, CGL is equivalent to NLS. So we study how the solutions for CGL converge to the corresponding solutions for NLS when we take the limit as $a \to 0$ and $b \to 0$. We call this procedure *the inviscid limit*, in the sense that we make the *viscosity a* tend to zero. We give the convergence rate for difference of the solutions.

CGL and NLS have been investigated by many authors (see, e.g., [6,7,13–15] and references therein for CGL, e.g., [5,9,10,16] and references therein for NLS). We are interested especially in the case that the initial data for these equations are not smooth. In that case, roughly speaking, we may think that the convergence rate of solutions depends on the properties of the solutions for NLS mainly, since the solutions for CGL possess better smoothing effect. More precisely, the convergence rate for the parameter a depends on the regularity of the solutions for NLS, and the one for parameter b depends on their integrability. We will observe those properties by making use of the fundamental solutions for CGL and NLS, namely the integral forms of the solutions for them. Thus we can see the smoothing effect of the solutions directly. We consider only the $\mathbf{R}^n$ case.

The problem on the inviscid limit have been considered for Navier–Stokes equation (see [8]), for Burgers' equation (see [3]), etc. There are a few results on this problem for CGL. In [19], L^2 convergence rate $O(a^{1/2}) + O(b^{p/2(p+1)})$ for H^1 data was shown (for notation, see the bottom of this section). L^{p+1} and H^1 convergence rates with stronger regular data were also studied. In [1], weak H^1 convergence for H^1 data was shown. L^2 convergence rate $O(a) + O(b)$ with H^s, $s > n/2$, data were also studied. In these papers energy method was used for the proof. So the high regularity was necessary for the initial data. There are the related results obtained by using monotonicity method. In [13,15], the estimate for the following nonlinear Schrödinger equation with monotone nonlinearity instead of (1.2) was obtained:

$$\text{NLS'}: \quad \partial_t v = i\nu\Delta v - b|v|^{p-1}v, \quad v(0) = v_0.$$

After having completed this paper, the authors found the Wang's results [17]. In his argument, the fundamental solution was used, however, since $a\Delta u$ included in the principal part of (1.1) was treated as the perturbation term of (1.2), his results seem to be based on the energy method essentially.

As mentioned above, in this paper, we study this problem by different way, namely we investigate the integral forms for CGL and NLS by using the Strichartz estimate. The Strichartz estimate for NLS, which displays the smoothing effect of the Schrödinger equation, is well known (see [5,9–11,16] and references therein). We construct the similar one for CGL to study this problem in the uniform framework for the difference of solutions between two equations. Using this estimate, we derive the existence results for CGL which are corresponding to the one of NLS, and consider the inviscid limit problem.

We make use of some techniques from the paper [12] which studies on the nonrelativistic limit of the nonlinear Klein–Gordon equation (NLKG), that is, on how the solutions for NLKG converge to the corresponding solutions for NLS when the speed of light tends to infinity.

We use the following notation. For any $1 \leqslant r \leqslant \infty$, $s \in \mathbb{R}$, we denote L^r for Lebesgue space and $H^{s,r}$ for Sobolev space with the norm

$$\|u\|_{H^{s,r}} = \left\|(1-\Delta)^{s/2}u\right\|_{L^r}.$$

We abbreviate $H^{s,2}$ to H^s. We use $\hat{}$ to denote the Fourier transform with respect to the space variable. $o(\cdot)$ and $O(\cdot)$ mean *small order* and *large order*, respectively. We denote the dual exponent to p by p', namely $1/p + 1/p' = 1$. Set $0 < a, b < 1$. Occasionally we abbreviate "$\leqslant C$" to "$\lesssim$" where C is a positive constant independent of a and b.

2. Preliminaries and results

Since we consider the convergence of the solutions of (1.1) to the solution of (1.2), we treat the solutions of (1.1) as the sequence depending on (a, b).

Definition 1. We denote the initial data by $u_{0\varepsilon}$ and solution for (1.1) which is obtained above by u_ε, where $\varepsilon = (a, b)$, and we promise that $\varepsilon \to 0$ means $a \to 0$ and $b \to 0$.

Remark. Even if we regard the solutions of (1.1) as the sequence depending on (ν, μ) and consider the convergence to the solution of the nonlinear heat equation

$$\text{NLH:} \quad \partial_t v = a\Delta v - b|v|^{p-1}v, \quad v(0) = v_0,$$

we also obtain the same convergence rate as in the following theorems on (1.2). Indeed, the heat kernel and the Schrödinger kernel have the same decay rates as $t \to \infty$ (see, e.g., [4,18]).

We set $f(w) = |w|^{p-1}w$ for simplicity. Then (1.1) and (1.2) can be rewritten in the form of integral equations respectively as follows:

$$\text{CGL}_{\text{int}}: \quad u_\varepsilon(t) = U_a(t)u_{0\varepsilon} - (b + i\mu)\int_0^t U_a(t-s) f\big(u_\varepsilon(s)\big)\, ds$$

$$\text{with} \quad U_a(t) = e^{(a+i\nu)\Delta t}, \tag{2.1}$$

$$\text{NLS}_{\text{int}}: \quad v(t) = V(t)v_0 - i\mu\int_0^t V(t-s) f\big(v(s)\big)\, ds$$

$$\text{with} \quad V(t) = e^{i\nu\Delta t}. \tag{2.2}$$

We study the evolution operator $U_a(t)$ and $V(t)$ and gain the Strichartz type estimate. Simultaneously, we obtain the existence results for CGL with the same regularity for NLS case. We collect the existence results for NLS (see, e.g., [5,9–11,16]). Especially, we introduce only the subcritical case that the existence of the global solution in time to (1.2) is guaranteed.

Definition 2. The pair (q, r) of real numbers is said to be admissible if $1/r + 2/nq = 1/2$ with $2 \leqslant r \leqslant 2n/(n-2)$ when $n \geqslant 3$, $2 \leqslant r < \infty$ when $n = 2$, $2 \leqslant r \leqslant \infty$ when $n = 1$.

Proposition 3. *Let* $1 \leqslant p < 1+4/n$*. Let* $v_0 \in L^2(\mathbb{R}^n)$*. Then there exists a unique solution* v *for* (1.2) *satisfying*

$$v \in C\bigl([0,\infty); L^2(\mathbb{R}^n)\bigr) \cap L^{\mathsf{q}}_{\mathrm{loc}}\bigl(0,\infty; L^{p+1}(\mathbb{R}^n)\bigr),$$

where $(\mathsf{q}, p+1)$ *with* $\mathsf{q} = 4(p+1)/(np-n)$ *is an admissible pair. Moreover, for any* $T < \infty$*, there exists a constant* $M = M(T) < \infty$ *such that*

$$\|v\|_{L^q(0,T;L^r(\mathbb{R}^n))} < M$$

for any admissible pair (q,r)*.*

Furthermore, the solution v *for* (1.2) *satisfies, for* $0 \leqslant t < \infty$*,*

$$Q\bigl(v(t)\bigr) = Q(v_0),$$

where $Q(w) = \|w\|_{L^2(\mathbb{R}^n)}$*.*

Proposition 4. *Let* $\mu > 0$*. Let* $1 \leqslant p < \infty$ *for* $n = 1, 2$*,* $1 \leqslant p < (n+2)/(n-2)$ *for* $n \geqslant 3$*. Let* $v_0 \in H^s(\mathbb{R}^n)$*,* $s = 1, 2$*. Then there exists a unique solution* v *for* (1.2) *satisfying*

$$v \in C\bigl([0,\infty]; H^s(\mathbb{R}^n)\bigr) \cap C^1\bigl([0,\infty]; H^{s-2}(\mathbb{R}^n)\bigr).$$

Moreover, for any $T < \infty$*, there exists a constant* $M = M(T) < \infty$ *such that*

$$\|v\|_{L^q(0,T;H^{s,r}(\mathbb{R}^n))} < M$$

for any admissible pair (q,r)*. In addition, if* $s = 2$*, then there exists a constant* $M' = M'(T) < \infty$ *such that*

$$\|\partial_t v\|_{L^q(0,T;L^r(\mathbb{R}^n))} < M'.$$

Furthermore, the solution v *for* (1.2) *satisfies, for* $0 \leqslant t < \infty$*,*

$$Q\bigl(v(t)\bigr) = Q(v_0), \qquad E\bigl(v(t)\bigr) = E(v_0),$$

where $Q(w)$ *as above and*

$$E(w) = \frac{\nu}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^n)} + \frac{\mu}{p+1}\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}.$$

Remark. When $\mu < 0$ and $1 \leqslant p < 1 + 4/n$, Proposition 4 also holds true.

Remark. When p is critical, i.e., $p = 1 + 4/n$ in Proposition 3 and $p = (n+2)/(n-2)$ for $n \geqslant 3$ in Proposition 4, it is well known that, under the additional assumption that the norm of the initial data is small, there exists a unique time global solution of (1.2) (see, e.g., [5]).

We have the well-posedness theorems of (1.1) corresponding to the above theorems for (1.2).

Theorem 5. *Let* ε *be fixed. Let* $1 \leqslant p < 1 + 4/n$*. Let* $u_{0\varepsilon} \in L^2(\mathbb{R}^n)$*. Then there exists a unique solution* u_ε *for* (1.1) *satisfying*

$$u_\varepsilon \in C\bigl([0,\infty); L^2(\mathbb{R}^n)\bigr) \cap L^{\mathsf{q}}_{\mathrm{loc}}\bigl(0,\infty; L^{p+1}(\mathbb{R}^n)\bigr),$$

where $(\mathfrak{q}, p+1)$ *as above. Moreover, for any* $T < \infty$*, there exists a constant* $M = M(T) < \infty$ *such that*

$$\|u_\varepsilon\|_{L^q(0,T;L^r(\mathbb{R}^n))} < M$$

for any admissible pair (q, r)*.*

Furthermore, the solution u_ε *for* (1.1) *satisfies, for* $0 \leqslant t < \infty$*,*

$$Q\big(u_\varepsilon(t)\big) < Q(u_{0\varepsilon}),$$

where $Q(w) = \|w\|_{L^2(\mathbb{R}^n)}$*.*

Theorem 6. *Let* ε *be fixed and* $\mu > 0$*. Let* $1 \leqslant p < \infty$ *for* $n = 1, 2$, $1 \leqslant p < (n+2)/(n-2)$ *for* $n \geqslant 3$*. Let* $u_{0\varepsilon} \in H^s(\mathbb{R}^n)$*,* $s = 1, 2$*. Then there exists a unique solution* u_ε *for* (1.1) *satisfying*

$$u_\varepsilon \in C\big([0, \infty); H^s(\mathbb{R}^n)\big) \cap C^1\big([0, \infty); H^{s-2}(\mathbb{R}^n)\big).$$

Moreover, for any $T < \infty$*, there exists a constant* $M = M(T) < \infty$ *such that*

$$\|u_\varepsilon\|_{L^q(0,T;H^{s,r}(\mathbb{R}^n))} < M$$

for any admissible pair (q, r)*. In addition, if* $s = 2$*, then there exists a constant* $M' = M'(T) < \infty$ *such that*

$$\|\partial_t u_\varepsilon\|_{L^q(0,T;L^r(\mathbb{R}^n))} < M'.$$

Furthermore, the solution u_ε *for* (1.1) *satisfies, for* $0 \leqslant t < \infty$*,*

$$Q\big(u_\varepsilon(t)\big) \leqslant Q(u_{0\varepsilon}), \qquad E\big(u_\varepsilon(t)\big) \leqslant E(u_{0\varepsilon}),$$

where $Q(w)$ *and* $E(w)$ *as above.*

Remark. Similarly to NLS cases, when $\mu < 0$ and $1 \leqslant p < 1 + 4/n$, Theorem 6 also holds true.

Remark. Similarly to NLS cases, when p is critical, i.e., $p = 1 + 4/n$ in Theorem 5 and $p = (n+2)/(n-2)$ for $n \geqslant 3$ in Theorem 6, we can prove that, under the additional assumption that the norm of the initial data is small, there exists a unique time global solution of (1.1).

Now we give the convergence theorems, which are main results in this paper. We note that we will take the limit $\varepsilon \to 0$. We set the initial data in L^2, H^1 and H^2, respectively.

Theorem 7. *Let assumptions as in Theorem* 5 *be satisfied. Let* $u_{0\varepsilon} = v_0 \in L^2$*. Then, for any* $T > 0$*,*

$$\|u_\varepsilon - v\|_{L^q(0,T;L^r)} \leqslant o(1) + O(b)$$

for any admissible pair (q, r)*. In particular,*

$$\|u_\varepsilon - v\|_{L^\infty(0,T;L^2)} \leqslant o(1) + O(b). \tag{2.3}$$

Theorem 8. *Let assumptions as in Theorem* 6 *be satisfied. Let* $u_{0\varepsilon} = v_0 \in H^1$. *Then, for any* $T > 0$ *and any admissible pair* (q, r),

$$\|u_\varepsilon - v\|_{L^q(0,T;L^r)} \leqslant \rho(a) + O(b),$$

where $\rho(a) = o(\sqrt{a})$; *more precisely* $\|\rho(a)/\sqrt{a}\|_{l^2L^\infty} < \infty$, *where*

$$\big\|g(a)\big\|_{l^2L^\infty}^2 = \sum_{j\in\mathbb{N}} \sup_{2^{-j}\leqslant a\leqslant 2^{-j+1}} \big|g(a)\big|^2.$$

In particular,

$$\|u_\varepsilon - v\|_{L^\infty(0,T;L^2)} \leqslant \rho(a) + O(b). \tag{2.4}$$

Combining the above two theorems, we obtain the strong convergence in H^1.

Corollary 9. *Let assumptions as in Theorem* 8 *be satisfied. Then, for any* $T > 0$,

$$\|u_\varepsilon - v\|_{L^\infty(0,T;H^1)} \leqslant o(1) + O(b).$$

Remark. In [1], Bechouche and Jüngel proved the weak H^1 convergence of u_ε to v. From this property, we can prove the strong H^1 convergence by using the conservation law of Q and E of the solution for NLS immediately, without the use of the fundamental solutions. In this case, we can obtain only the estimate

$$\|u_\varepsilon - v\|_{L^\infty(0,T;L^2)} \leqslant O\big(\sqrt{a}\big) + O(b).$$

Corollary 10. *Let assumptions as in Theorem* 8 *be satisfied. Then, for any* $T > 0$ *and any* r *with* $2 < r < 2n/(n-2)$,

$$\|u_\varepsilon - v\|_{L^\infty(0,T;L^r)} \leqslant o(1) + O(b).$$

Theorem 11. *Let assumptions as in Theorem* 6 *be satisfied. And let* $u_{0\varepsilon} = v_0 \in H^2$. *Then, for any* $T > 0$ *and any admissible pair* (q, r),

$$\|u_\varepsilon - v\|_{L^q(0,T;L^r)} \leqslant O(a) + O(b).$$

In particular,

$$\|u_\varepsilon - v\|_{L^\infty(0,T;L^2)} \leqslant O(a) + O(b). \tag{2.5}$$

Remark. From the same argument of the proof of Theorem 1.2 in [12], we can prove the optimality for (2.3), (2.4) and (2.5).

Remark. When there exist only a local solution in time to (1.1) and (1.2), the above convergence theorems hold locally uniformly on $[0, T^*)$ where T^* is the maximal existence time of the solution of (1.2). In fact, denoting by T_ε^* the maximal existence time of the solution of (1.1), we can prove

$$\liminf_{\varepsilon\to 0} T_\varepsilon^* \geqslant T^*$$

(see [12]). This fact can be applied to the following cases:

(i) p is critical, i.e., $p = 1 + 4/n$ in Theorem 7 and $p = (n+2)/(n-2)$ for $n \geqslant 3$ in Theorem 8, and there is no size restriction on the initial data.
(ii) $\mu < 0$, $1 + 4/n \leqslant p < \infty$ for $n = 1, 2$, $1 + 4/n \leqslant p \leqslant (n+2)/(n-2)$ for $n \geqslant 3$ and $u_{0\varepsilon} = v_0 \in H^1$.
(iii) $\mu > 0$, $(n+2)/(n-2) \leqslant p < \infty$ for $n = 3, 4$, $(n+2)/(n-2) \leqslant p \leqslant n/(n-4)$ for $n \geqslant 5$ and $u_{0\varepsilon} = v_0 \in H^2$.
(iv) $\mu < 0$, $1 + 4/n \leqslant p < \infty$ for $1 \leqslant n \leqslant 4$, $1 + 4/n \leqslant p \leqslant n/(n-4)$ for $n \geqslant 5$ and $u_{0\varepsilon} = v_0 \in H^2$; etc.

3. Proof

Firstly in this section, we consider CGL. We give the existence and the uniqueness of the solution for (1.1). We study it on the integral form (2.1), and so it is important to study the operator $U_a(t)$. Now we give the Strichartz type estimate for CGL.

Lemma 12. *Let $u \in L^2$ and $f \in L^{q_3'}(0, T; L^{r_3'})$ with some $0 < T \leqslant \infty$ and some admissible pair (q_3, r_3). For any admissible pair (q_j, r_j), $j = 1, 2$, the following estimates hold*:

$$\left\| U_a(t)u \right\|_{L^{q_1}(0,\infty;L^{r_1})} \lesssim \|u\|_{L^2}, \tag{3.1}$$

$$\left\| \int_0^t U_a(t-s) f(s)\, ds \right\|_{L^{q_2}(0,T;L^{r_2})} \lesssim \|f\|_{L^{q_3'}(0,T;L^{r_3'})}, \tag{3.2}$$

where the boundedness "$\lesssim$" is independent of u, f, T and a.

Remark. Indeed it is well known that the LHS of (3.2) with the norm of more general spaces except for the endpoint (see, e.g., [4,7,18]). Since we are interested in the difference between the solutions for CGL and for NLS, we restrict our attention to the spaces labeled the admissible pairs including the endpoints to which the solution for NLS belongs.

Proof of Lemma 12. It is well known that for $2 < q_1 \leqslant \infty$, (3.1) holds. For the endpoint $(q_1, r_1) = (2, 2n/(n-2))$ with $n \geqslant 3$, we apply the result of Keel and Tao [11]. By the contraction properties of $e^{at\Delta}$, we have

$$\left\| e^{(a+i\nu)t\Delta} u_0 \right\|_{2n/(n-2)} \leqslant \left\| e^{it\Delta} u_0 \right\|_{2n/(n-2)}$$

for any $t > 0$ and $n \geqslant 3$. Hence we have

$$\left\| U_a(\cdot)u_0 \right\|_{L^2(0,\infty;L^{2n/(n-2)})} \leqslant \left\| V(\cdot)u_0 \right\|_{L^2(0,\infty;L^{2n/(n-2)})} \lesssim \|u_0\|_2,$$

by the endpoint Strichartz estimate for the free Schrödinger operator.

We prove (3.2) for any admissible pairs (q_j, r_j), $j = 2, 3$. When $q_2 \neq q_3'$ or $r_2 \neq r_3'$, since it is clearly that $0 < 1/q_3' - 1/q_2 = 1 - (n/2)(1/r_3' - 1/r_2) < 1$, it is well known that (3.2) holds. When $q_2 = q_3' = 2$, that is, $r_2 = 2n/(n-2)$ and $r_3' = 2n/(n+2)$ following to the proof of the inhomogeneous endpoint Strichartz estimate in Keel and Tao [11], we can prove

$$\left\| \int_0^t U_a(t-s) f(s)\,ds \right\|_{L^2(0,T;L^{2n/(n-2)})} \leqslant C \|f\|_{L^2(0,T;L^{2n/(n+2)})}.$$

Note that $e^{at\Delta}$, $t > 0$, is a contraction map on L^r, $1 \leqslant r < \infty$. When $r_2 = r_3' = 2$, that is, $q_2 = \infty$ and $q_3' = 1$, we have for any $t \in [0, T]$

$$\begin{aligned}
\left\| \int_0^t U_a(t-s) f(s)\,ds \right\|_{L^2}^2 &= \left(\int_0^t e^{(a+i\nu)\Delta(t-s)} f(s)\,ds, \int_0^t e^{(a+i\nu)\Delta(t-s')} f(s')\,ds' \right) \\
&= \int_0^t \int_0^t \left(e^{a\Delta(2t-s-s')} f(s), e^{i\nu\Delta(s-s')} f(s') \right) ds\,ds' \\
&\leqslant \int_0^t \int_0^t \left\| e^{a\Delta(2t-s-s')} f(s) \right\|_{L^2} \left\| e^{i\nu\Delta(s-s')} f(s') \right\|_{L^2} ds\,ds' \\
&\lesssim \int_0^t \left\| f(s) \right\|_{L^2} \left(\int_0^t \left\| f(s') \right\|_{L^2} ds' \right) ds \lesssim \|f\|_{L^1(0,T;L^2)}^2.
\end{aligned}$$

We note $2t - s - s' \geqslant 0$. □

Proof of Theorem 5. We employ the contraction mapping principle. The following arguments are well known for NLS. Let $\mathfrak{r} = p + 1$. We set the space $X_T^p = L^\infty(0, T; L^2) \cap L^{\mathfrak{q}}(0, T; L^{\mathfrak{r}})$ with norm

$$\|w\|_{X_T^p} = \|w\|_{L^\infty(0,T;L^2)} + \|w\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})},$$

and the space $X_T^{p'} = L^1(0, T; L^2) + L^{\mathfrak{q}'}(0, T; L^{\mathfrak{r}'})$ with norm

$$\|w\|_{X_T^{p'}} = \inf\left\{ \|w_1\|_{L_T^1(L^2)} + \|w_2\|_{L_T^{\mathfrak{q}'}(L^{\mathfrak{r}'})}\colon w = w_1 + w_2 \right\}.$$

We start from (2.1). By (3.1) and (3.2), we have

$$\|u_\varepsilon\|_{X_T^p} \lesssim \|u_{0\varepsilon}\|_{L^2} + \left\| f(u_\varepsilon) \right\|_{L^{\mathfrak{q}'}(0,T;L^{\mathfrak{r}'})} \leqslant \|u_{0\varepsilon}\|_{L^2} + T^\theta \|u_\varepsilon\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})}^p,$$

where $\theta = 1 - n(p-1)/4 > 0$. Similarly we have contraction properties as

$$\begin{aligned}
\left\| u_\varepsilon^1 - u_\varepsilon^2 \right\|_{X_T^p} &\lesssim \left\| f\left(u_\varepsilon^1\right) - f\left(u_\varepsilon^2\right) \right\|_{L^{\mathfrak{q}'}(0,T;L^{\mathfrak{r}'})} \\
&\leqslant T^\theta \left(\left\| u_\varepsilon^1 \right\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})} + \left\| u_\varepsilon^2 \right\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})} \right)^{p-1} \left\| u_\varepsilon^1 - u_\varepsilon^2 \right\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})}.
\end{aligned}$$

Therefore we can close the contraction map for sufficiently small T which is independent of a and b, but depends only on $\|u_{0\varepsilon}\|_{L^2}$, and we obtain a unique time local solution for (1.1). Since we have $\|u_\varepsilon(t)\|_2 \leqslant \|u_{0\varepsilon}\|_2$ for any $t > 0$, we can continue this solution time globally. □

Proof of Theorem 6. First we construct H^1-solution. We set the spaces $Y_T^p = \{u \in X_T^p \mid \nabla u \in X_T^p\}$ with norm

$$\|u\|_{Y_T^p} = \|u\|_{X_T^p} + \|\nabla u\|_{X_T^p},$$

and $Y_T^p = \{u \in X_T^{p'} \mid \nabla f \in X_T^{p'}\}$ with norm

$$\|f\|_{Y_T^{p'}} = \|f\|_{X_T^{p'}} + \|\nabla f\|_{X_T^{p'}}.$$

We can verify that U_a is a bounded operator from H^1 to Y_T^p and A_a is a bounded operator from $Y_T^{p'}$ to Y_T^p, immediately, where $(A_a f)(t) = \int_0^t T_a(t-s) f(s)\,ds$. And it is easily seen that, if $u \in Y_T^p$, then $f(u) \in Y_T^{p'}$. We note that $L^{p+1} \supset H^1$ by Sobolev embedding theorem is used in this proof (see [10]). Therefore we can close the contraction map for sufficiently small T which is independent of a and b, but depends only on $\|u_{0\varepsilon}\|_{H^1}$ as above. On the other hand, we have $E(u_\varepsilon(t)) \leqslant E(u_{0\varepsilon})$ and $\|u_\varepsilon(t)\|_2 \leqslant \|u_{0\varepsilon}\|_2$ for any $t > 0$. This implies that $\|u_\varepsilon(t)\|_{H^1}$ is bounded for any $t > 0$, therefore we can continue this solution time globally.

For H^2-solution, we set the space $Z_T^p = \{u \in L^\infty(0,T; H^2) \mid \partial_t u \in X_T^p\}$ with norm

$$\|u\|_{X_T^p} = \|u\|_{X_T^p} + \|\partial u\|_{X_T^p} + \|\Delta u\|_{L^\infty(0,T;L^2)}.$$

We can verify that U_a is a bounded operator from H^2 to Z_T^p, immediately. $\|A_a f\|_{Z_T^p}$ is finite if $f \in L^\infty(0,T; L^2)$ and $\partial_t f \in X_T^{p'}$, since $f(u) \in L^\infty(0,T; L^2)$ for $u \in Z_T^{p'}$. We note that $L^{2p} \supset H^2$ by Sobolev embedding theorem is used in this proof (see [10]). Therefore, we can close the contraction map for sufficiently small T which is independent of a and b, but depends only on $\|u_{0\varepsilon}\|_{H^2}$ as above. On the other hand, we can prove that $\|u_\varepsilon(t)\|_{H^2}$ is bounded for any $t > 0$ (see [9,10] for (1.2)), therefore we can continue this solution time globally. $\square$

When we investigate the difference between the solutions, the following proposition give us the important information.

Proposition 13. *Let $s_1 \geqslant s_2 \geqslant 0$ and $\phi \in H^{s_1}$. Then the following estimate holds for any* $T > 0$:

$$\big\|\big(U_a(\cdot) - V(\cdot)\big)\phi\big\|_{L^\infty(0,T;H^{s_2})} \leqslant \Theta_a^{s_1,s_2},$$

where

$$\Theta_a^{s_1,s_2} = \begin{cases} o(a^{(s_1-s_2)/2}) & \text{if } 0 \leqslant s_1 - s_2 < 2, \\ O(a) & \text{if } 2 \leqslant s_1 - s_2. \end{cases}$$

We note that, in the case of $s_1 = s_2$, $\Theta_a^{s_1,s_2} = o(1)$.

We can prove this proposition by using the Fourier transform with respect to the space variable easily.

We prove the main theorems.

Proof of Theorem 7. We now turn on investigation of the inviscid limit from CGL to NLS. Subtracting (2.2) from (2.1),

$$
\begin{aligned}
u_\varepsilon(t) - v(t) = \big(U_a(t) - V(t)\big)v_0 - i\mu \int_0^t \big(U_a(t-s) - V(t-s)\big) f\big(v(s)\big)\, ds \\
- b\int_0^t U_a(t-s) f\big(u_\varepsilon(s)\big)\, ds \\
- i\mu \int_0^t U_a(t-s)\big(f\big(u_\varepsilon(s)\big) - f\big(v(s)\big)\big)\, ds \\
= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}
$$

We investigate $\|I_j\|_{L^q(0,T;L^r)}$, $j = 1, 2, 3, 4$, for any admissible pair (q, r), respectively.

For the estimate of I_1, we set the approximation $v_0^j \in C_0^\infty(\mathbb{R}^n)$ of v_0 such that

$$\big\|v_0 - v_0^j\big\|_{L^2} \to 0, \quad \text{as } j \to \infty.$$

Then we have for any admissible (q, r)

$$
\begin{aligned}
\big\|\big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big)v_0\big\|_{L^q(0,T;L^r)} &\leqslant \big\|e^{(a+i\nu)\Delta t}\big(v_0 - v_0^j\big)\big\|_{L^q(0,T;L^r)} \\
&\quad + \big\|e^{i\nu\Delta t}\big(v_0 - v_0^j\big)\big\|_{L^q(0,T;L^r)} + \big\|\big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big)v_0^j\big\|_{L^q(0,T;L^r)}.
\end{aligned}
$$

It is clear that the first and the second terms tend to 0 as $j \to \infty$. And for the third term, we have

$$
\begin{aligned}
\big\|\big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big)v_0^j\big\|_{L^r} &= \left\| \int_0^t a\Delta e^{a\Delta s} e^{i\nu\Delta t} v_0^j\, ds \right\|_{L^r} \\
&\leqslant a\int_0^t \big\|e^{a\Delta s}e^{i\nu\Delta t}\Delta v_0^j\big\|_{L^r}\, ds \leqslant aT\big\|e^{i\nu\Delta t}\Delta v_0^j\big\|_{L^r} \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|\big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big)v_0^j\big\|_{L^q(0,T;L^r)} &\leqslant aT\big\|e^{i\nu\Delta t}\Delta v_0^j\big\|_{L^q(0,T;L^r)} \\
&\lesssim aT\big\|\Delta v_0^j\big\|_{L^2} \to 0, \qquad (3.4)
\end{aligned}
$$

as $a \to 0$. Hence we have $\|I_1\|_{L^q(0,T;L^r)} \to 0$ as $a \to 0$.

To consider I_2, recall that $f(v) \in L^{\mathfrak{q}'}(0, T; L^{\mathfrak{r}'})$ (see [10]). Note that $C_0^\infty(\mathbb{R}^{n+1})$ is dense in $L^{\mathfrak{q}'}(0, T; L^{\mathfrak{r}'})$, so we have $\|I_2\|_{L^q(0,T;L^r)} \to 0$ as $a \to 0$ from the same manner on I_1.

We know

$$\|I_3\|_{L^q(0,T;L^r)} \lesssim bT^\theta \|u_\varepsilon\|^p_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})} \lesssim b.$$

For the estimate of I_4, we divide the proof into the two cases. When $2 \leqslant r \leqslant p+1$, we have

$$\begin{aligned}\|I_4\|_{L^q(0,T;L^r)} &\leqslant \|I_4\|_{X_T^p} \lesssim \big\| f(u_\varepsilon) - f(v)\big\|_{L^{\mathfrak{q}'}(0,T;L^{\mathfrak{r}'})} \\ &\leqslant T^\theta \big(\|u_\varepsilon\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})} + \|v\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})}\big)^{p-1} \|u_\varepsilon - v\|_{L^{\mathfrak{q}}(0,T;L^{\mathfrak{r}})} \\ &\lesssim T^\theta \|u_\varepsilon - v\|_{X_T^p}.\end{aligned}$$

Collecting the above estimates, we have for some constant $C > 0$

$$\|u_\varepsilon - v\|_{L^q(0,T;L^r)} \leqslant \|u_\varepsilon - v\|_{X_T^p} \leqslant o(1) + O(b) + CT^\theta \|u_\varepsilon - v\|_{X_T^p},$$

as $\varepsilon \to 0$. Therefore for sufficiently small T, we obtain the desired result which is extended to any $T > 0$ by repeating the above arguments. For any $p+1 \leqslant r \leqslant 2n/(n-2)$, there exists some $2 \leqslant \gamma \leqslant 2n/(n-2)$ such that $1/2 \leqslant (p-1)/\gamma + 1/r \equiv 1/\alpha \leqslant 1/2 + 1/n$. Setting $2/\beta = n(1/2 + 2/n - 1/\alpha)$, we have

$$\begin{aligned}\|I_4\|_{L^q(0,T;L^r)} &\lesssim \big\| f(u_\varepsilon) - f(v)\big\|_{L^\beta(0,T;L^\alpha)} \\ &\leqslant T^\theta \big(\|u_\varepsilon\|_{L^\rho(0,T;L^\gamma)} + \|v\|_{L^\rho(0,T;L^\gamma)}\big)^{p-1} \|u_\varepsilon - v\|_{L^q(0,T;L^r)} \\ &\lesssim T^\theta \|u_\varepsilon - v\|_{L^q(0,T;L^r)},\end{aligned}$$

where $2/\rho = n(1/2 - 1/\gamma)$; note that $\theta = 1 - n(p-1)/4$. Using this estimate, we can obtain the desired result directly. $\square$

Proof of Theorem 8. We have already estimated $\|I_j\|_{L^q(0,T;L^r)}$, $j = 3, 4$, above. Note that $H^1 \subset L^{p+1}$ for $1 \leqslant p < (n+2)/(n-2)$. We investigate only the rests which include the parameter a. For I_1, we estimate for any admissible pair (q, r) (see [2])

$$\begin{aligned}&\big\| \big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big) v_0 \big\|_{L^q(0,T;L^r)} \\ &\quad\lesssim \left(\sum_{j=0}^{\infty} \big\| \big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big) \varphi_j * v_0 \big\|_{L^q(0,T;L^r)}^2 \right)^{1/2},\end{aligned} \tag{3.5}$$

where $\{\varphi_j\}_{-\infty}^{\infty}$ is the Littlewood–Paley dyadic decomposition on $\mathbb{R}^n$, namely $\{\hat{\varphi}_j\} \subset C_0^\infty$, $\operatorname{supp} \hat{\varphi}_j \subset \{\xi;\ 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$, $0 \leqslant \hat{\varphi}_j \leqslant 1$, and $\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1$ on $\mathbb{R}^n/\{0\}$. We rewrite φ_0 for ψ such that $\hat{\psi} = 1 - \sum_{j=1}^{\infty} \hat{\varphi}_j$.

Using (3.1) on the summand in the right hand side of (3.5) directly and on the form

$$\big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big) \varphi_j * v_0 = a \int_0^t e^{a\Delta s + i\nu\Delta t} \Delta \varphi_j * v_0 \, ds,$$

we gain

$$\big\| \big(e^{(a+i\nu)\Delta t} - e^{i\nu\Delta t}\big) \varphi_j * v_0 \big\|_{L^q(0,T;L^r)} \lesssim \min(1, a2^{2j}) \|\varphi_j * v_0\|_{L^2}.$$

We set $g(a) = (\sum_{j=0}^{\infty} \min(1, a^2 2^{4j}) ||\varphi_j * v_0\|_{L^2}^2)^{1/2}$, which is estimated by

$$\begin{aligned}\left\|g(a)/\sqrt{a}\right\|_{l^2L^\infty}^2 &= \sum_{k=0}^{\infty} \sup_{2^{-k}\leqslant a\leqslant 2^{-k+1}} \sum_{j=0}^{\infty} \min\left(a^{-1}, a2^{4j}\right)\|\varphi_j * v_0\|_{L^2}^2 \\ &\lesssim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \min\left(2^{k-2j}, 2^{-k+2j}\right)2^{2j}\|\varphi_j * v_0\|_{L^2}^2 \\ &\lesssim \sum_{j=0}^{\infty} 2^{2j}\|\varphi_j * v_0\|_{L^2}^2 \lesssim \|v_0\|_{H^1}^2.\end{aligned}$$

So we may write $\|I_1\|_{L^q(0,T;L^r)} \leqslant \rho(a)$, where $\rho(a) = o(\sqrt{a})$ as $a \to 0$. Since $f(v) \in L^{q'}(0,T;H^{1,r'})$ (see [10]), we have from the same manner on I_1

$$\|I_2\|_{L^q(0,T;L^r)} \leqslant \rho(a),$$

as $a \to 0$. □

Proof of Corollary 9. From the arguments in the proofs of Theorems 5 and 7, we have for any $T < \infty$

$$\left\|\nabla(u_\varepsilon - v)\right\|_{L^\infty(0,T;L^2)} \leqslant o(1) + O(b),$$

as $a \to 0$ (see the proof of Theorem II$'$ in [10]). Therefore combining (2.4) with this, we obtain the strong convergence in H^1. □

Proof of Corollary 10. Applying Sobolev embedding theorem to Corollary 9, we have the desired estimate. □

Proof of Theorem 11. If suffices to estimate only I_1 and I_2 as defined in the proof of Theorem 7. Since $v_0 \in H^2$, we can estimate I_1 as in (3.3) and (3.4). Therefore we have $\|I_1\|_{L^q(0,T;L^r)} \leqslant O(a)$ as $a \to 0$. Similarly we have $\|I_2\|_{L^q(0,T;L^r)} \leqslant O(a)$ as $a \to 0$. □

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